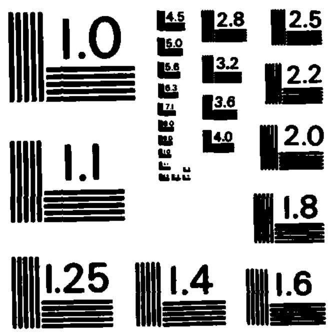
AD-A120 965 ON STEADY VORTEX FLOW IN TWO DIMENSIONS I(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER B TURKINGTON JUL 82 MRC-TSR-2406 DAAG29-80-C-0041 1/1

UNCLASSIFIED F/G 20/4 NL

END

FILMED

DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

MRC Technical Summary Report # 2406

ON STEADY VORTEX FLOW IN TWO DIMENSIONS, I

Bruce Turkington

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

July 1982

(Received March 25, 1982)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

ON STEADY VORTEX FLOW IN TWO DIMENSIONS, I

Bruce Turkington*

Technical Summary Report #2406
July 1982

ABSTRACT

We study a family of steady flows of an ideal fluid in a bounded domain $D \subseteq R^2$ using a variational method. Adapting an approach due to Arnold, we characterize dynamically possible flows as constrained extremals of the kinetic energy considered as a functional of the vorticity $\omega(x)$, $x \in D$. We restrict our attention to solutions having the special form:

$$\omega = \omega_\lambda = \lambda \, I_{\{\psi>0\}} \, , \int_D \omega_\lambda dx = 1$$

where λ is a free parameter and ψ (the streamfunction) is defined by $-\Delta\psi = \omega$ in D, $\psi = -\mu$ on ∂D (μ is an undetermined positive constant). A simple proof of existence of solutions is provided by a direct variational method. The main results concern the singular limit $\lambda \to \infty$ (which implies $\mu \to \infty$). We prove that $\omega_\lambda \to \delta_{x^*}$, the Dirac delta measure at $x^* \in D$, in the distributional sense, and we characterize the point x^* in terms of the geometry of D. Further, we show that supp ω_λ tends asymptotically (in a strong sense) to an infinitesimal disc about x^*. This follows since appropriately scaled version of the solutions ω_λ tend to the unique radial solution of a corresponding limit problem. In fluid dynamical terms, the classical point vortex is obtained as the limit of vortices of finite cross-section (of the Rankine type).

AMS (MOS) Subject Classifications: 76C05, 35R35.

Key Words: vorticity, variational inequality, asymptotic analysis, singular limit, point vortex, free boundary problem.

Work Unit Number 1 (Applied Analysis)

*Department of Mathematics, Northwestern University, Evanston, IL, 60201

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041, and in part by the National Science Foundation, Grant No. MCS-8002927.

SIGNIFICANCE AND EXPLANATION

The purpose of this paper (and its sequel) is to give a mathematically rigorous - as well as physically natural - discussion of certain steady solutions of the Euler dynamical equations for an ideal, two-dimensional fluid. The flows considered have a prescribed distribution of vorticity ω = curl u (u devotes the velocity field), and are separated into regions where $\omega = 0$ and $\omega > 0$. The shape and position of the "vortex core" (region where $\omega > 0$) for a flow satisfying the dynamical requirements is then determined by the geometry of the fluid domain (assumed bounded here). Solutions of the fluid dynamical equations are most conveniently characterized by a variational principle which involves finding an extreme value for the kinetic energy of the flow subject to certain natural constraints. This approach permits a precise analysis of the properties of solutions to be carried out in a unified manner. In this respect, special emphasis is placed upon deriving the (classical) point vortex as the limit of solutions with concentrated vorticity.

The occurence of vortices (or "eddies") in real fluid flows is, of course, well known. The traditional literature is most often restricted to idealizations in which (for two dimensions) the vorticity is concentrated (as Dirac deltas) at a finite number of points. Also, numerical simulations of vortex flows are frequently based upon this discrete vortex approximation. The present work, however, is devoted to a simple model problem for which, first, the appropriate mathematical theory is developed without the above mentioned restriction, and second, the (asymptotic) nature of the traditional idealization is analysed qualitatively. It should also be remarked that some of the methods developed here for steady flows can be adapted to the corresponding time-dependent flows.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON STEADY VORTEX FLOW IN TWO DIMENSIONS, I

Bruce Turkington*

In this paper we study a certain variational problem whose extremals represent steady solutions of the Euler dynamical equations for an ideal fluid in two dimensions. We focus our attention on a special family of such flows, each of which has a prescribed distribution of vorticity and is separated into regions of zero and of positive vorticity; we refer to flows of this type as steady vortex flows. Using an adaptation of the variational approach of Arnold, we obtain dynamically possible steady vortex flows by extremizing the kinetic energy functional subject to some appropriate constraints. This particular variational method is well suited both to prove the

*Department of Mathematics, Northwestern University, Evanston, IL 60201

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041, and in part by the National Science Foundation, Grant No. MCS-8002927.

existence of solutions and to determine the qualitative properties of these solutions. We first give a simple proof of existence using a direct (variational) method, and compute the somewhat novel variational conditions satisfied by extremals. We then turn to our principal results which concern the asymptotic analysis of solutions (each representing a vortex of finite cross-section) in a certain singular limit (the limit being a classical point vortex). In the latter analysis we feel that an advantage is achieved here over other approaches to this and related problems since sharp results on the asymptotic nature of solutions are obtained by relatively elementary means.

We confine ourselves in the present paper (Part I) to a very simple flow geometry : the fluid domain is bounded and the flow is everywhere tangential on the boundary. In a subsequent paper (Part II) we shall apply the results developed here in the simple case to other more complicated situations. In particular, we shall consider steady vortex pairs and, especially, steady vortical wakes behind a symmetric obstacle in a uniform stream. Axisymmetric analogues (such a vortex rings) can also be studied using similar methods.

§1. VARIATIONAL PROBLEM

Let $D \subseteq R^2$ be a bounded and simply-connected

domain with a smooth boundary, ∂D. Throughout the sequel we shall assume for convenience that diam $D = 1$; this is achieved after appropriately scaling the spatial variables, which we shall write as $x = (x_1, x_2) \in R^2$.

We consider a steady flow of an ideal fluid of unit density in the (fluid) domain D. The velocity field $u(x) = (u_1(x), u_2(x))$ and the pressure $p(x)$ are required to satisfy the Euler equations:

$$\nabla\cdot u = 0 \quad \text{in} \quad D \quad \text{and} \quad \nu\cdot u = 0 \quad \text{on} \quad \partial D \tag{1.1}$$

$$u\cdot\nabla u = -\nabla p \quad \text{in} \quad D\,, \tag{1.2}$$

where ν is the exterior unit normal on ∂D. The kinematical conditions (1.1) are satisfied whenever there exists a stream function $\psi = \psi(x)$, $x \in D$, in terms of which the velocity field can be represented as

$$u = J\nabla\psi = (\psi_{x_2}, -\psi_{x_1}) \quad \text{in} \quad D \quad \text{and} \tag{1.3}$$
$$\psi = \text{const. on} \quad \partial D;$$

here and in the sequel, $J(a_1, a_2) = (-a_2, a_1)$ denotes clockwise rotation through $\pi/2$. We now express the dynamical condition (1.2) in a weak form, which will, in turn, lead to its variational characterization. The vorticity $\omega = \omega(x)$, $x \in D$, is defined to be

$$\omega = u_{2,x_1} - u_{1,x_2} = -\Delta\psi \quad \text{in} \quad D\,. \tag{1.4}$$

It follows by a well-known calculation that then (1.2) becomes

$$\omega\nabla\psi = -\nabla(p + \tfrac{1}{2}|\nabla\psi|^2) \quad \text{in} \quad D\,. \tag{1.5}$$

We recall that a vector field $a = a(x)$ can be expressed as a gradient $a = \nabla b$ for some function $b = b(x)$ if and only if

$$\int_D a \cdot J\nabla\phi dx = 0 \quad \text{for all} \quad \phi \in C_0^\infty(D) \ .$$

Applying this to (1.5) we have that a pressure p exists if and only if

$$(1.6) \qquad \int_D \omega\partial(\psi,\phi)dx = 0 \quad \text{for all} \quad \phi \in C_0^\infty(D) \ ;$$

here we use the shorthand notation

$$\partial(\psi,\phi) = \nabla\psi \cdot J\nabla\phi = \psi_{x_1}\phi_{x_2} - \psi_{x_2}\phi_{x_1} \ .$$

We let the Green function for $-\Delta$ in D with homogeneous Dirichlet data on ∂D be written

$$(1.7) g(x,x') = \frac{1}{2\pi} \log|x-x'|^{-1} - h(x,x') \quad x,x' \in D \ .$$

The corresponding Green operator we denote by

$$(1.8) \qquad G\omega(x) = \int_D g(x,x')\omega(x')dx' \ .$$

We also define (for later use) the Kirchhoff-Routh function to be

$$(1.9) \qquad H(x) = \tfrac{1}{2} h(x,x) \qquad x \in D \ .$$

In terms of these, equation (1.4) inverts to become

$$(1.10) \qquad \psi = G\omega - \mu \ , \quad \mu = -\psi|_{\partial D} \ .$$

Returning to (1.6) the dynamical requirement is now expressed as a condition on ω alone:

$$(1.11) \int_D \omega\partial(G\omega,\phi)\, dx = 0 \qquad \text{for all} \quad \phi \in C_0^\infty(D) \ .$$

Thus, the fluid dynamical problem (1.1,1.2) is reduced to the problem: find ω such that (1.11) holds. We proceed by characterizing such ω as the constrained

extremals of a certain energy functional.

The kinetic energy of the flow is

$$E = \tfrac{1}{2}\int_D |u|^2 dx = \tfrac{1}{2}\int_D |\nabla G\omega|^2 dx \ ;$$

so E defines a functional of ω, which upon integration by parts becomes

$$E(\omega) = \tfrac{1}{2}\int_D \omega(x) G\omega(x) dx$$

(1.12)

$$= \tfrac{1}{2}\int_D \int_D g(x,x')\omega(x)\omega(x') dx dx' \ .$$

We now construct a family of variations, $\omega^{(t)}$, of ω for any given ϕ as in (1.11). Let $x = \xi_t(y)$ denote the solution of the initial value problem

$$\frac{dx}{dt} = J\nabla\phi(x) \quad , \quad x(0) = y \in D$$

in a small interval $t \in (-T,T)$, say. The transformations $y \to \xi_t(y)$ then form a (smooth) one-parameter family of area-preserving diffeomorphisms of D. The variations are defined by

$$\omega^{(t)}(x) = \omega(\xi_t^{-1}(x)) \qquad x \in D \ .$$

A calculation employing the symmetry property $g(x,x') = g(x',x)$ then yields the first variation:

$$E(\omega^{(t)}) = \tfrac{1}{2}\int_D \int_D g(x,x')\omega(\xi_t^{-1}(x))\omega(\xi_t^{-1}(x')) dx dx'$$

$$= \tfrac{1}{2}\int_D \int_D g(\xi_t(y),\xi_t(y'))\omega(y)\omega(y') dy dy'$$

$$= E(\omega) + t\int_D \omega\partial(G\omega,\phi) dx + o(t)$$

as $t \to 0$. Consequently, (1.11) is equivalent to

$$0 = \frac{d}{dt} E(\omega^{(t)}) \,|_{t=0} \,. \tag{1.13}$$

Therefore, we have shown that (local) extremals of $E(\omega)$ over any class of admissible functions ω chosen wide enough to contain all variations of the form $\omega^{(t)}$ will yield solutions of the original problem (1.1, 1.2) - that is, dynamically possible steady vortex flows in D.

A natural choice for the class of admissible functions for $E(\omega)$ is the class of (measure-theoretic) rearrangements of a given function ω_0; that is, all those ω satisfying

$$\begin{aligned} &\text{meas}\,\{x \in D : \omega(x) > \lambda\} \\ &\qquad = \text{meas}\,\{x \in D : \omega_0(x) > \lambda\} \quad \text{for all} \quad \lambda \in R \,. \end{aligned}$$

At least formally, any (local) extremal for $E(\omega)$ over this class of rearrangements yields a solution ω of (1.11) having the prescribed measure distribution belonging to ω_0. A variational principle of this form has been proposed by Arnold [1] (Appendix 2) and certain criteria for the stability of steady flows have been derived. Also Benjamin [2] has discussed an existence and stability theory for steady vortex rings in this context. In the present work we choose to consider instead a wider class of admissible functions in order to both facilitate the (rigorous) analysis and

obtain solutions having an especially simple form. We wish to model a steady flow in D which is separated into an irrotational region and a region of constant (positive) vorticity; we do not permit vortex sheets interior to D. To this end we let $K_\lambda(D)$, the class of admissible functions, consist of all those $0 \leq \omega \in L^\infty(D)$ satisfying the constraints

$$(1.14) \qquad \int_D \omega(x)dx = 1$$

$$(1.15) \qquad \operatorname*{ess\,sup}_{x \in D} \omega(x) \leq \lambda$$

for a parameter $\lambda > (\text{meas } D)^{-1}$. The circulation in (1.14) is normalized to be 1 after scaling the dependent variable ω; λ plays the role of a (vortex) strength parameter.

The existence result proved in §2 is the following: there exists an absolute maximizer, $\omega = \omega_\lambda \in K_\lambda(D)$, for E over $K_\lambda(D)$ and it has the special form

$$(1.16) \qquad \omega = \lambda I_\Omega, \ \Omega = \{x \in D : \psi(x) > 0\}$$

where ψ is defined by (1.10) and the constant $\mu > 0$ is uniquely determined by ω; I_A denotes the characteristic function of the set A. The statement (1.16) represents the strong form of the variational conditions for the extremal ω (it implies the weak form (1.6)), and μ arises as a Lagrange multiplier

for the constraint (1.14). The streamfunction ψ then solves a nonlinear free boundary problem which can be expressed compactly as an elliptic equation with discontinuous nonlinearity:

$$(1.17) \qquad -\Delta\psi = \lambda I_{\{\psi>0\}} \quad \text{in} \quad D;$$

we note that the velocity field $u = J\nabla\psi$ is continuous across the free boundary, $\partial\Omega$. In other words, when ω takes the special form (1.16) the general condition (1.6) reduces simply to the requirement that $\partial\Omega$ be a streamline (for ψ).

There is no corresponding uniqueness result for solutions, or even for absolute maximizers of the form (1.16). Indeed, for certain geometrics nonuniqueness can be proved using the asymptotic results in §3-4.

The main body of analysis occurs in §3-4 where the asymptotic nature of maximizers ω_λ as $\lambda \to \infty$ is determined. We expect to obtain in the limit a steady point vortex; indeed we show that

$$(1.18) \qquad \omega_\lambda(x) \to \delta(x-X^*) \quad \text{as} \quad \lambda \to \infty$$

where the convergence is in the sense of distributions and $\delta(z)$ denotes the unit (Dirac) delta measure concentrated at $z = 0$. Furthermore, the location $X^* \in D$ is determined (by the geometry of D) according to

$$(1.19) \quad H(X^*) = \min_{x \in D} H(x) \qquad \text{(recall (1.9))} .$$

Strictly speaking, as X^* is not necessarily uniquely determined the statement (1.18) may hold only for a sequence $\lambda = \lambda_j \to \infty$. In §3 the basic asymptotic estimate for large λ -from which all else follows - is proved; it states that

$$\text{(1.20)} \qquad \operatorname{diam}(\operatorname{supp} \omega_\lambda) < R\varepsilon \quad (\lambda\pi\varepsilon^2=1)$$

for a positive constant $R > 1$ (independent of λ). In §4 we establish the sharp limiting forms of ω_λ and $\partial\Omega_\lambda$. To do so we introduce the center of vorticity

$$X_\lambda = \int_D x\, \omega_\lambda(x)dx ,$$

and the scaled version or ("blow-up") of ω_λ:

$$\begin{aligned} \zeta_\lambda(y) &= \lambda^{-1}\omega_\lambda(X_\lambda + \varepsilon y) && (\lambda\pi\varepsilon^2=1) \\ &= I_{A_\lambda}(y) && y \in B_R(0), \end{aligned}$$

where $A_\lambda \subseteq B_R(0)$ is the scaled version of Ω_λ. We prove that a unique radial limit is approached as $\lambda \to \infty$:

$$\zeta_\lambda \to I_{B_1(0)} \qquad \text{weakly star in } L^\infty(D)$$

$$\partial A_\lambda \to \partial B_1(0) \qquad C^1 \text{ sense as curves .}$$

These results depend upon some symmetry lemmas in potential theory given at the beginning of §4.

We remark that the condition (1.19) determining X^* is anticipated by the classical theory of a point vortex moving through an ideal fluid (otherwise irrotational) within a bounded domain. According to the theory developed by Kirchhoff and Routh, a point

vortex of unit circulation concentrated at $X(t) \in D$ (t is time) moves along a trajectory determined by the system

(1.21) $$\frac{dX}{dt} = -J\nabla H(X) ,$$

evidently a Hamiltonian system with one degree of freedom. Now we see why the limit as $\lambda \to \infty$ of steady flows induced by ω_λ should tend to a steady point vortex at X^* satisfying (1.19); indeed, X^* is an equilibrium point of the Hamiltonian system (1.21) since $\nabla H(X^*) = 0$. For a full discussion of the classical theory we refer the reader to Lin [12]; see also the recent work of Richardson [15].

It is possible to study a closely related class of variational problems in which solutions of the form (1.16) are replaced by solutions of the following form:

$$\omega = \lambda(\psi^+)^\beta \quad 0 < \beta < \infty \quad (\psi \text{ as above}) .$$

The appropriate class of admissible functions is $K_\infty(D) \subseteq L^p(D)$ which is obtained by removing the constraint (1.15), and the appropriate functional is

$$E_\beta(\omega) = E(\omega) - \int_D \frac{\lambda}{p}\left[\frac{\omega(x)}{\lambda}\right]^p dx$$

for $p = 1 + 1/\beta$; $\lambda > 0$ is prescribed. The existence of an absolute maximizer $\omega = \omega_{\lambda,\beta}$ for E_β over $K_\infty(D)$ can be proved using a direct method similar to that given in §2; general existence theorems of this type appear in Berestycki and Brezis [3]. Asymptotic

results for $\omega_{\lambda,\beta}$ as $\lambda \to \infty$ can also be shown by natural (though not entirely obvious) extensions of the techniques developed in §3-4. In particular, the estimate (1.20) for large λ holds for arbitrary $0 < \beta < \infty$ (fixed); we will not present the proof here. This asymptotic estimate has been proved already for $\beta = 1$ by Caffarelli and Friedman [6], and for $\beta > 1$ by Berger and Fraenkel [5]; however these authors begin with a different variational characterization of solutions from the above. Their proofs (fundamentally the same in both) are entirely different from our proof (§3). First the connectedness of $\Omega_{\lambda,\beta} = \{\psi > 0\}$ must be established (this seems to require $\beta > 1$ unless special symmetry of the domain D is assumed), and then the (potential-theoretic) capacity of $\Omega_{\lambda,\beta}$ in D is estimated; this yields (1.20) since the capacity can be related to the diameter of $\Omega_{\lambda,\beta}$. The techniques given in §3 seem preferable, however, because (i) they are simpler and (ii) they are more closely allied with the techiques needed to study analogous time-dependent flows with concentrated vorticity; we intend to pursue this latter topic elsewhere. Finally, we remark that the functional E_β represents a penalization of the energy functional E for small β, and in fact the solutions ω_λ (as in (1.16)) can be obtained as the

limits of the (penalized) solutions $\omega_{\lambda,\beta}$ as $\beta \to 0^+$. Accordingly, alternate proofs of all the theorems of §2 can be based on this procedure.

Related papers not already cited include - Goldstik [9], Keady and Norbury [11], Keady [10], Gallouët [8]; in the context of vortex pairs and/or rings - Berger and Fraenkel [4], Norbury [14], Friedman and Turkington [7]. The applied literature is referenced in Saffman and Baker [16].

§2. EXISTENCE

We recall from §1 the definitions:

$$K_\lambda(D) = \{\omega \in L^\infty(D) : \int_D \omega(x)dx = 1 , \quad 0 \leq \omega(x) \leq \lambda \text{ a.e. } x \in D\} \tag{2.1}$$

$$E(\omega) = \tfrac{1}{2} \int_D \int_D g(x,x')\omega(x)\omega(x')dxdx' . \tag{2.2}$$

We assume throughout the sequel that

$$\lambda > (\text{meas } D)^{-1} . \tag{2.3}$$

An absolute maximum for E over $K_\lambda(D)$ is now found by a simple direct method.

Theorem 2.1. There exists $\omega = \omega_\lambda \in K_\lambda(D)$ such that

$$E(\omega) = \max_{\tilde{\omega} \in K_\lambda(D)} E(\tilde{\omega}) . \tag{2.4}$$

Proof. We shall often use the obvious estimate (recall

diam D = 1)

(2.5) $g(x,x') \leq \frac{1}{2\pi} \log |x - x'|^{-1} \qquad x,x' \in D$.

That E is bounded above on $K_\lambda(D)$ is evident from:

$$
\begin{aligned}
2E(\tilde{\omega}) &\leq \sup_{x \in D} G\tilde{\omega}(x) \\
&\leq \sup_{x \in D} \frac{1}{2\pi} \int_D \log |x - x'|^{-1}\tilde{\omega}(x')dx' \\
&\leq \frac{\lambda}{2\pi} \int_{|y| < \varepsilon} \log |y|^{-1}dy \qquad (\lambda\pi\varepsilon^2=1) \\
&= \frac{1}{2\pi} \left(\log \frac{1}{\varepsilon} + \tfrac{1}{2}\right) = C(\lambda) .
\end{aligned}
$$

Thus there is a sequence $\omega_j \in K_\lambda(D)$ such that

$$\lim_{j\to\infty} E(\omega_j) = \sup_{\tilde{\omega} \in K_\lambda(D)} E(\tilde{\omega}) < \infty .$$

As $K_\lambda(D)$ is clearly a compact subset of $L^\infty(D)$ in the weak star topology we can extract a subsequence of ω_j (call it again ω_j) such that

$$\omega_j \to \omega \in K_\lambda(D) \text{ weakly star in } L^\infty(D) .$$

Now it remains to prove $E(\omega_j) \to E(\omega)$; but this follows easily since $g(x,x') \in L^1(DXD)$ and $\omega_j(x)\omega_j(x') \to \omega(x)\omega(x')$ weakly star in $L^\infty(DXD)$. This completes the proof.

The calculations of §1 imply that any maximizer ω yields a dynamically possible steady flow; we state this fact next. We note that, by standard potential theory, $G\omega \in H^2(D)\cap H_0^1(D)$ and $G\omega \in C^{1,\alpha}(\bar{D})$ for every

$0 < \alpha < 1$.

Corollary 2.2. Whenever $\omega \in K_\lambda$ satisfies (2.4) then there holds

(2.6) $\int_D \omega \partial(G\omega, \phi)dx = 0$ for all $\phi \in C_0^\infty(D)$.

Integrating by parts formally in (2.6) we get $\partial(G\omega, \omega) = 0$. This means that there is a functional dependence F $(\nabla F \neq 0)$ such that $F(G\omega, \omega) = 0$ identically. Condition (2.6) may be said to express the functional dependence between $G\omega$ and ω "weakly" therefore. We now give this dependence explicitly.

Corollary 2.3. Whenever $\omega \in K_\lambda(D)$ satisfies (2.4) then

(2.7) $\omega = \lambda I_\Omega$ a.e. in D, $\Omega = \{x \in D\colon G\omega(x) > \mu\}$

for a constant $\mu > 0$ uniquely determined by ω; I_Ω denotes the characteristic function of Ω.

Proof. We consider a family of variations of ω different from that used in the derivation of (2.6); namely, we define

$$\omega_{(s)}(x) = \omega(x) + s\,[z_0(x) - z_1(x)] \qquad s > 0$$

for arbitrary $z_0, z_1 \in L^\infty(D)$ satisfying

$$\int_D z_0(x)dx = \int_D z_1(x)dx$$

$$z_0, z_1 \geq 0 \quad \text{a.e. in } D$$

$$z_0 = 0 \quad \text{a.e. in } D\backslash\{\omega < \lambda - \delta\}$$

$$z_1 = 0 \quad \text{a.e. in } D\backslash\{\omega > \delta\}$$

for some $\delta > 0$. When $s > 0$ is sufficiently small (depending on δ, $\|z_0\|_\infty$, $\|z_1\|_\infty$) we have $\omega_{(s)} \in K_\lambda(D)$, and hence we conclude that

$$0 \geq \frac{d}{ds} E(\omega_{(s)}) \big|_{s=0^+}$$
$$= \int_D z_0(x)G\omega(x)dx - \int_D z_1(x)G\omega(x)dx \; .$$

It is easy to see that the latter inequality holds for arbitrary z_0, z_1 as above and arbitrary $\delta > 0$ only if

(2.8) $$\operatorname*{ess\,sup}_{\omega(x)<\lambda} G\omega(x) \leq \operatorname*{ess\,inf}_{\omega(x)>0} G\omega(x) \; .$$

The continuity of $G\omega$ implies easily that strict inequality cannot hold in (2.8); hence we may define

(2.9) $$\mu = \operatorname*{ess\,sup}_{\omega(x)<\lambda} G\omega(x) = \operatorname*{ess\,inf}_{\omega(x)>0} G\omega(x) \; .$$

From this it is now clear that $\omega = \lambda$ a.e. in $\{G\omega > \mu\}$ and $\omega = 0$ a.e. in $\{G\omega < \mu\}$. Therefore, in order to conclude the desired representation (2.7) it suffices to show that $\omega = 0$ a.e. in $S = \{G\omega = \mu\}$ (as we have not excluded meas $S > 0$). To prove this we observe that both $G\omega$ and $\nabla G\omega$ when considered as functions of either x_1 or x_2 alone are absolutely continuous on almost all lines (segments in D)

parallel to the coordinate axes. Thus, as almost all points of S are points of density of S along such lines, it follows that the pointwise first and second partial derivatives of $G\omega$ vanish a.e. in S. Hence $\omega = -\Delta G\omega = 0$ a.e. in S, as required. This reasoning is taken from Morrey [13] Theorem 3.2.2(c). Finally we notice that $\mu > 0$ since otherwise $\omega = \lambda$ a.e. in D, contradicting hypothesis (2.3). This completes the proof.

We leave the general question of the regularity of the free boundary $\partial\{\psi > 0\} \subseteq \{\psi = 0\}$ unresolved here. All subsequent results of this paper depend upon (2.7) only. In §4, however, we do prove that the free boundary is a simple closed C^1 curve if λ is large enough. For arbitrary λ we conjecture that the set of irregular free boundary points (where $\nabla\psi = 0$) is finite or has one-dimensional Hausdorff measure zero, but this appears to be an open problem.

§3. ASYMPTOTIC ESTIMATE

This section is devoted to the proof of estimate (1.20) which is basic to the asymptotic analysis of solutions as $\lambda \to \infty$. We suppose throughout that ω satisfies (2.4), and we write $\psi = G\omega - \mu$ according to (2.7). For any such ω we define

(3.1) $T(\omega) = \frac{1}{2} \int_D |\nabla\psi^+|^2 dx \qquad (\psi^+ = \max\{\psi, 0\})$;

T is the kinetic energy of the vortex core

$\Omega = \{\psi > 0\}$.

We prepare the proof of the main theorem by two short lemmas. We write $C, C_1, C_2, \ldots,$ for positive constants independent of λ.

<u>Lemma 3.1</u>. There holds the lower bound:

$$E(\omega) > \frac{1}{4\pi} \log \frac{1}{\varepsilon} - C_1 \quad (\lambda\pi\varepsilon^2 = 1) \ . \tag{3.2}$$

<u>Proof</u>. We consider for large λ

$$\hat{\omega} = \lambda I_{B_\varepsilon(\hat{X})} \in K_\lambda(D) \tag{3.3}$$

where $\hat{X} \in D$ is chosen such that $H(\hat{X}) = \min_{x \in D} H(x)$ (the reader will verify that $H(x) \to +\infty$ as $x \to \partial D$). Then we may estimate

$$E(\omega) > E(\hat{\omega})$$

$$= \frac{1}{4\pi} \int_D \int_D \log|x-x'|^{-1} \hat{\omega}(x)\hat{\omega}(x')dxdx' - H(\hat{X}) + o(1)$$

$$> \frac{1}{4\pi} \log \frac{1}{2\varepsilon} - H(\hat{X}) + o(1)$$

as $\lambda \to \infty$. From this (3.2) clearly follows.

<u>Lemma 3.2</u>. There holds the upper bound:

$$T(\omega) = \frac{1}{2} \int_D \psi\omega dx < C_2 \tag{3.4}$$

<u>Proof</u>. Since $\psi = -\mu < 0$ on ∂D we may integrate by parts to get

$$2T(\omega) = \int_D \nabla\psi^+ \cdot \nabla\psi dx = \int_D \psi^+ \omega dx = \int_D \psi\omega dx,$$

using (2.7) in the last equality. Furthermore, we have

$$\int_D \psi\omega dx = \lambda \int_D \psi^+ dx \le \lambda \; (\text{meas } \Omega)^{1/2} \{ \int_D (\psi^+)^2 \}^{1/2} .$$

Thus, by applying the Sobolev inequality to ψ^+

$$\{ \int_D (\psi^+)^2 dx\}^{1/2} \le C \int_D |\nabla\psi^+| dx,$$

we then obtain

$$2T(\omega) \le C\lambda(\text{meas } \Omega)^{1/2} \int_D |\nabla\psi^+| dx$$

$$\le C\lambda(\text{meas } \Omega)\{ \int_D |\nabla\psi^+|^2 dx\}^{1/2}$$

$$= C\,[2T(\omega)]^{1/2} ;$$

this establishes (3.4).

<u>Theorem 3.3</u>. There is a constant $R > 1$ (independent of λ) such that

(3.5) $\quad \text{diam (supp } \omega) \le R\varepsilon \qquad (\lambda\pi\varepsilon^2 = 1)$.

<u>Proof</u>. We estimate the parameter μ using the obvious identity

(3.6) $\quad T(\omega) = E(\omega) - \frac{1}{2}\mu$;

then Lemmas 3.1 and 3.2 combine to give

(3.7) $\quad \mu \ge \frac{1}{2\pi} \log \frac{1}{\varepsilon} - C_3$.

Now let $x \in \operatorname{supp} \omega$ be fixed arbitrarily. Then $G\omega(x) > \mu$, and so we find (recalling (2.5))

$$\frac{1}{2\pi} \log \frac{1}{\varepsilon} - C_3$$

$$< \int_D g(x,x')\omega(x')dx' < \frac{1}{2\pi} \int_D \log |x-x'|^{-1}\omega(x')dx' .$$

Equivalently, we write (for any $R > 1$)

$$- 2\pi C_3 < \int_D \log \frac{\varepsilon}{|x-x'|} \omega(x')dx'$$

$$= \int_{B_{R\varepsilon}(x)} + \int_{D\setminus B_{R\varepsilon}(x)} \log \frac{\varepsilon}{|x-x'|} \omega(x')dx' .$$

We observe now that

$$\int_{B_{R\varepsilon}(x)} \log \frac{\varepsilon}{|x-x'|} \omega(x')dx' < \lambda \int_{|y|<\varepsilon} \log \frac{\varepsilon}{|y|} dy = \tfrac{1}{2} .$$

Therefore we get

$$-2\pi C_3 - \tfrac{1}{2} < \int_{D\setminus B_{R\varepsilon}(x)} \log \frac{\varepsilon}{|x-x'|} \omega(x')dx'$$

$$< (\log \frac{1}{R}) \int_{D\setminus B_{R\varepsilon}(x)} \omega(x')dx' ;$$

after manipulation this becomes

$$(3.8) \qquad \int_{D\setminus B_{R\varepsilon}(x)} \omega(x')dx' < C_4(\log R)^{-1} .$$

We now see that the desired estimate (3.5) follows immediately from inequality (3.8). Indeed, we claim that $\operatorname{diam} (\operatorname{supp} \omega) < 2R\varepsilon$ if R is fixed large enough to ensure that $C_4(\log R)^{-1} < \tfrac{1}{2}$. Otherwise there would exist $x^1, x^2 \in \operatorname{supp} \omega$ with the property that $B_{R\varepsilon}(x^1) \cap B_{R\varepsilon}(x^2) = \phi$ and so, by (3.8),

$$\int_D \omega(x')dx' \geq \int_{B_{R\varepsilon}(x^1)} \omega(x')dx' + \int_{B_{R\varepsilon}(x^2)} \omega(x')dx' > 1 ;$$

this contradicts the constraints for $\omega \in K_\lambda(D)$, and so the theorem is proved.

<u>Remark</u>. Of course, the constant R depends upon the domain; an examination of the proof shows that R can be taken in the form: $R = c \exp(kH^*)$ where $H^* = \min_D H$ and c,k are absolute positive constants.

§4. LIMITING BEHAVIOR

In this section we study the limiting form of the maximizers $\omega = \omega_\lambda$ supplied by Theorem 2.1 as $\lambda \to \infty$. We find that their appropriately scaled versions tend to a unique limiting function which is necessarily radially symmetric. To this end we first prove some symmetry lemmas in potential theory which are themselves of independent interest.

<u>Lemma 4.1</u>. Suppose that an open set A with $\bar{A} \subseteq B_R(0)$, $0 < R < \infty$, possesses the property that $A = \{x \in B_R(0) : V(x) > \gamma\}$ for some constant $\gamma = \gamma(A)$ where $V = V_A$ is the potential of A:

$$V(x) = \frac{1}{2\pi} \int_A \log |x-x'|^{-1} dx' . \tag{4.1}$$

Then A is necessarily a ball (and so V is radial).

Proof. We shall utilize a reflection (or "folding") argument very similar to an argument used by Serrin [17] for a different symmetry result in potential theory. In particular, we shall show that A is symmetric across a line x_1 = const.; this suffices to prove the radial symmetry of A since we may rotate coordinate axes arbitrarily without changing the conclusion.

We define the open sets, for $-R < t < R$,

$A_t = A \cap \{x : x_1 < t\}$, $A_t^* = \{x : (2t-x_1, x_2) \in A_t\}$;

that is, A_t^* is the reflection of A_t across the line $x_1 = t$. We then consider $s = \max\ \{t : A_t^* \subseteq A\}$ (it is easy to see that this maximum is achieved and that $-R < s < R$). We now claim that A is symmetric across the line $x_1 = s$; we intend to establish the claim by showing that

(4.2) $\quad$ meas $N = 0$ $\quad$ for $\quad N = A \setminus (A_s \cup A_s^*)$.

First we observe that there must exist a point $x^* \in \partial A \cap \partial A_s^*$, since otherwise $\overline{A}_s^* \subseteq A$ and hence $A_t^* \subseteq A$ for some $t > s$ (contradicting the maximality of s). We then write $x = (2s-x_1^*, x_2^*) \in \partial A_s$. We now proceed to consider the two distinct cases:

(1) $x_1 < s < x_1^*$, (2) $x_1 = s = x_1^*$. The reader may check by example that either case may occur.

Case 1. Since $x, x^* \in \partial A$ there holds

$$0 = V(x^*) - V(x) = \frac{1}{2\pi} \int_N \log \frac{|x-x'|}{|x^*-x'|}\, dx' ,$$

owing to the fact that the corresponding integrals over A_s and A_s^* cancel each other. But the above integrand is strictly positive on $N \setminus \{x_1 = s\}$ and so it is not possible that meas $N > 0$; thus (4.2) follows in this case.

Case 2. Whenever $x = x^* \in \partial A \cap \partial A_s^* \cap \{x_1 = s\}$ there holds

$$V_{x_1}(x) = \frac{1}{2\pi} \int_N |x - x'|^{-2}(x_1' - x_1)dx' ,$$

because again the corresponding integrals over A_s and A_s^* cancel each other. Thus if meas $N > 0$ then $V_{x_1}(x) > 0$ (as the integrand is strictly positive on $N \setminus \{x_1 = s\}$). Also, using the fact that now $\nabla V \neq 0$ at any such point $x = x^*$, we have by the implicit function theorem that ∂A is a C^1 curve in a neighborhood of $x = x^*$. But then it is not difficult to see that these facts together imply that there must exist $t > s$ such that $A_t^* \subseteq A$, a contradiction; thus (4.2) also follows in this case.

Lemma 4.2. Let the class K^* consist of all those $0 \le \zeta \in L^\infty(B_R(0))$, $1 < R < \infty$, satisfying the constraints

$$(4.3) \qquad \int_{B_R(0)} \zeta(x)dx = \pi , \qquad \operatorname*{ess\,sup}_{x \in B_R(0)} \zeta(x) \le 1 .$$

Let the functional F be defined by

$$(4.4) \qquad F(\zeta) = \frac{1}{4\pi} \int_{B_R(0)} \int_{B_R(0)} \log |x - x'|^{-1} \zeta(x)\zeta(x')dxdx' .$$

Then $\zeta^* = I_{B_1(0)}$ is the unique maximimizer of F over K^* for which

(4.5) $$\int_{B_R(0)} x\zeta(x)dx = 0 .$$

Proof. The methods of §2 apply here without change. We remark that the existence of a maximizer follows by the proof of Theorem 2.1. Furthermore, the proof of Corollary 2.3 shows that any maximizer ζ of F over K^* has the form

$$\zeta = I_A , \; A = \{x \in B_R(0) : V(x) > \gamma\}$$

for V given by (4.1) and some constant γ (determined by ζ as before). Therefore, Lemma 4.1 applies and we conclude that A is necessarily a ball of radius 1; obviously, (4.5) is imposed to determine the center of the ball uniquely.

Remark. Both of the preceding lemmas have natural generalizations to n dimensions which can be proved using the same methods. Also a wide class of kernels (defining the potential V and the functional F) can be permitted in place of the Newtonian kernel. We leave these extensions to the interested reader.

We now proceed to determine the limiting behavior of ω_λ as $\lambda \to \infty$. We define the center of vorticity to be

(4.6) $$X_\lambda = \int_D x\omega_\lambda(x)dx .$$

We now fix (for the remainder of the discussion) a sequence $\lambda = \lambda_j \to \infty$ such that there exists a limiting

center

(4.7) $$X_\lambda \to X^* \in \overline{D} \quad \text{as} \quad \lambda = \lambda_j \to \infty .$$

Theorem 4.3. Any X^* as in (4.7) satisfies

(4.8) $$H(X^*) = \min_{x \in D} H(x) .$$

Proof. Let $\hat{\omega} = \hat{\omega}_\lambda$ be defined by (3.3) with $H(\hat{X}) = \min_{x \in D} H(x)$. It is a consequence of Lemma 4.2 that then

$$\tfrac{1}{2} \int_D \int_D h(x,x')\omega_\lambda(x)\omega_\lambda(x')dxdx'$$

$$= \frac{1}{4\pi} \int_D \int_D \log |x-x'|^{-1}\omega_\lambda(x)\omega_\lambda(x')dxdx' - E(\omega_\lambda)$$

$$\leq \frac{1}{4\pi} \int_D \int_D \log |x-x'|^{-1}\hat{\omega}_\lambda(x)\hat{\omega}_\lambda(x')dxdx' - E(\omega_\lambda)$$

$$\leq \tfrac{1}{2} \int_D \int_D h(x,x')\hat{\omega}_\lambda(x)\hat{\omega}_\lambda(x')dxdx' ,$$

using simply $E(\omega_\lambda) \geq E(\hat{\omega}_\lambda)$ in the last inequality. Taking $\lambda \to \infty$ in the above we obtain, by virtue of the estimate (3.5), the inequality $H(X^*) \leq H(\hat{X})$; this is the required result (4.8).

We can now say that (due to (3.5)) as $\lambda = \lambda_j \to \infty$

(4.9) $$\omega_\lambda(x) \to \delta(x-X^*) \qquad \text{as distributions}$$

where $\delta(x-X^*)$ is the unit (Dirac) delta measure at $x = X^*$ given according to (4.8). The precise

asymptotic nature of ω_λ is expressed in terms of its scaled version in the following theorem.

<u>Theorem 4.4</u>. Let $\zeta_\lambda \in L^\infty(B_R(0))$ be defined by

(4.10) $$\zeta_\lambda(y) = \lambda^{-1}\omega_\lambda(X_\lambda+\varepsilon y) \quad (\lambda\pi\varepsilon^2=1)$$

for fixed R as in (3.5). Then as $\lambda \to \infty$

(4.11) $$\zeta_\lambda \to \zeta^* = I_{B_1(0)} \text{ weakly star in } L^\infty(B_R(0)) .$$

<u>Proof</u>. We retain the notation of Lemma 4.2. For any $\tilde{\zeta} \in K^*$, let $\tilde{\omega} \in K_\lambda(D)$ be defined as

$$\tilde{\omega}(x) = \begin{cases} \lambda\tilde{\zeta}(\varepsilon^{-1}(x-X_\lambda)) & x \in B_{R\varepsilon}(X_\lambda) \\ 0 & x \in D\setminus B_{R\varepsilon}(X_\lambda) . \end{cases}$$

A direct calculation then yields as $\lambda \to \infty$

(4.12) $$E(\tilde{\omega}) = \frac{1}{4\pi}\log\frac{1}{\varepsilon} + \frac{1}{\pi^2}F(\tilde{\zeta})$$

$$- \tfrac{1}{2}\int_D\int_D h(x,x')\omega(x)\omega(x')dxdx'$$

$$= \frac{1}{4\pi}\log\frac{1}{\varepsilon} + \frac{1}{\pi^2}F(\tilde{\zeta}) - H(X^*) + o(1) ;$$

and likewise this statement holds for $\tilde{\omega}$, $\tilde{\zeta}$ replaced by ω_λ, ζ_λ. But then $E(\tilde{\omega}) \le E(\omega_\lambda)$ implies that $F(\tilde{\zeta}) \le F(\zeta_\lambda) + o(1)$ as $\lambda \to \infty$. For any sequence $\lambda_m \to \infty$ there is a subsequence $\lambda_m' \to \infty$ such that $\zeta_{\lambda_m'} \to \zeta \in K^*$ weakly star in $L^\infty(B_R(0))$. Then $F(\zeta) = \lim_{m\to\infty} F(\zeta_{\lambda_m'}) \ge F(\tilde{\zeta})$ for all $\tilde{\zeta} \in K^*$, and so we

conclude by Lemma 4.2 that $\zeta = \zeta^* = I_{B_1(0)}$ (the construction (4.10) ensures condition (4.5)). Since this conclusion is independent of the sequence λ_m taken (by the uniqueness of ζ^*) we find that (4.11) holds whenever $\lambda \to \infty$.

The conclusion of the preceding theorem now implies the convergence of the scaled versions of the corresponding streamfunctions.

Theorem 4.5. Let $v_\lambda \in C^1(\overline{B}_{R'}(0))$ be defined by

(4.13) $$v_\lambda(y) = \pi\psi_\lambda(X_\lambda+\varepsilon y) \qquad (\lambda\pi\varepsilon^2 = 1)$$

for fixed R as in (3.5) and $R < R' < \infty$. Then as $\lambda \to \infty$

(4.14) $$v_\lambda \to v^* \quad \text{in} \quad C^1(\overline{B}_{R'}(0)$$

where v^* is given by

(4.15) $$v^*(y) = v^*(|y|) = \begin{cases} \frac{1}{4}\,(1-|y|^2) & 0 \leq |y| \leq 1 \\ \frac{1}{2}\,\log|y|^{-1} & 1 \leq |y| < \infty\,. \end{cases}$$

Remark. The limit function is also expressible as the potential

(4.16) $$v^*(y) = \frac{1}{2\pi}\int_{R^2} \log|y-y'|^{-1}\zeta^*(y')dy'\,.$$

We note that $v_\lambda,\ v^* \in H^2(B_{R'}(0)) \cap C^{1,\alpha}(\overline{B}_{R'}(0))$ for every $0 < \alpha < 1$, and $-\Delta v_\lambda = \zeta_\lambda$, $-\Delta v^* = \zeta^*$. We call v^* the (classical) Rankine streamfunction.

Proof. We define $V_\lambda \in C^1(R^2)$ by

$$V_\lambda(y) = \frac{1}{2\pi}\int_{R^2} \log |y-y'|^{-1}\zeta_\lambda(y')dy' .$$

Since $\operatorname{supp}\zeta_\lambda \subseteq \overline{B}_R(0)$ and $0 \leq \zeta_\lambda \leq 1$ a.e. in $B_R(0)$, we may apply standard potential theoretic estimates (see Morrey [13]) to conclude that

$$|\nabla V_\lambda(y)| \leq C$$

$$|\nabla V_\lambda(y^1)-\nabla V_\lambda(y^2)| \leq C|y^1-y^2| \log \left(1+\frac{2R'}{|y^1-y^2|}\right)$$

for all $y, y^1, y^2 \in \overline{B}_{R'}(0)$; C is a absolute constant. It follows that both families $\{V_\lambda\}$ and $\{\nabla V_\lambda\}$ are equicontinuous in $\overline{B}_{R'}(0)$. Since (4.11) implies the convergence $V_\lambda \to v^*$, $\nabla V_\lambda \to \nabla v^*$ pointwise in $B_{R'}(0)$, we find then that

(4.17) $V_\lambda \to v^*$ in $C^1(\overline{B}_{R'}(0))$ as $\lambda \to \infty$.

Let $D_\lambda = \{y \in R^2 : X_\lambda + \varepsilon y \in D\}$ denote the scaled version of D. Then v_λ is defined on D_λ and $v_\lambda|_{\partial D_\lambda} = \pi\psi_\lambda|_{\partial D} = -\pi\mu_\lambda$. We observe that by the identity (3.6) and the established upper and lower estimates for $E(\omega_\lambda)$ (along with the bound for $T(\omega_\lambda)$) there holds

$$\mu_\lambda = \frac{1}{2\pi} \log \frac{1}{\varepsilon} + O(1) \qquad \text{as } \lambda \to \infty .$$

Thus, since $d/\varepsilon \leq |y| \leq 1/\varepsilon$ for all $y \in \partial D_\lambda$ if $d = \frac{1}{2}\operatorname{dist}(X^*,\partial D) \leq \operatorname{dist}(X_\lambda,\partial D)$, we conclude that

$$v_\lambda(y) = \tfrac{1}{2}\log |y|^{-1} + O(1) = V_\lambda(y) + O(1) \quad \text{as } \lambda \to \infty$$

for all $y \in \partial D_\lambda$. Now since we have $\Delta(v_\lambda - V_\lambda) = 0$ in D_λ and $|v_\lambda - V_\lambda| \leq C$ (a constant independent of λ), we apply the interior gradient

estimate for harmonic functions to get

$$\sup_{y\in\overline{B}_{R'}(0)} |\nabla v_\lambda(y) - \nabla V_\lambda(y)| < C\varepsilon .$$

Therefore, $v_\lambda - V_\lambda \to c^* =$ const. in $C^1(\overline{B}_{R'}(0))$; now recalling (4.17) we find that $v_\lambda \to v^* + c^*$ in $C^1(\overline{B}_{R'}(0))$. Finally, we see that $c^* = 0$ since meas $\{v_\lambda > 0\} =$ meas $\{v^* > 0\} = \pi$ by virtue of the fact that $\zeta_\lambda = I_{\{v_\lambda > 0\}} \in K^*$.

<u>Corollary 4.6</u>. Let $A_\lambda = \{y \in B_{R'}(0) : v_\lambda(y) > 0\}$. Then A_λ tends asymptotically to the unit ball as $\lambda \to \infty$ in the sense that

(4.18) $A_\lambda = \{y = (r\cos\theta, r\sin\theta) : 0 < r < a_\lambda(\theta)\}$

where $a_\lambda \in C^1[0,2\pi]$ is a periodic function and

$$a_\lambda \to 1,\ \frac{da_\lambda}{d\theta} \to 0 \quad \text{uniformly in } [0,2\pi].$$

<u>Proof</u>. This is an immediate consequence of Theorem 4.5 by the implicit function theorem.

We end with some expansions.

<u>Corollary 4.7</u>. The following asymptotic expansions hold as $\lambda \to \infty$:

(4.19) $\quad E(\omega_\lambda) = \frac{1}{4\pi}\left(\log\frac{1}{\varepsilon} + \frac{1}{4}\right) - H(X^*) + o(1)$

(4.20) $$T(\omega_\lambda) = \frac{1}{16\pi} + o(1)$$

(4.21) $$\mu_\lambda = \frac{1}{2\pi} \log \frac{1}{\varepsilon} - 2H(X^*) + o(1) .$$

Proof. Derivation of (4.19) : Applying (4.12) to ω_λ and using $F(\zeta_\lambda) \to F(\zeta^*)$ we see that it suffices to calculate the value of $F(\zeta^*)$; this is given by

$$F(\zeta^*) = \frac{1}{2} \int_{B_1(0)} v^*(y)dy = \frac{\pi}{16} .$$

Derivation of (4.20) : Changing variables we have

$$2T(\omega_\lambda) = \frac{1}{\pi^2} \int_{B_R(0)} |\nabla v_\lambda^+|^2 dy \to \frac{1}{\pi^2} \int_{B_1(0)} |\nabla v^*|^2 dy = \frac{1}{8\pi} .$$

Derivation of (4.21): Using (3.2) this is immediate.

ACKNOWLEDGEMENTS

The author would like to thank Nicholas Korevaar and William Ziemer for some very helpful conversations.

REFERENCES

1. Arnold, V.I., Mathematical methods of classical mechanics. Springer Verlag, 1978.

2. Benjamin, T.B., The alliance of practical and analytical insights into the nonlinear problems of fluid mechanics. Applications of methods of functional analysis to problems of mechanics, Lecture Notes in Mathematics no. 503, Springer Verlag, 1976; pp. 8-29.

3. Berestycki, H. and Brezis, H., On a free boundary problem arising in plasma physics. Nonlinear Anal. 4 (1980), 415-436.

4. Berger, M.S. and Fraenkel, L.E., A global theory of steady vortex rings in an ideal fluid. Acta. Math. 132 (1974), 14-151.

5. Berger, M.S. and Fraenkel, L.E., Nonlinear desingularization in certain free-boundary problems. Comm. Math. Phys. 77 (1980), 149-172.

6. Caffarelli, L. and Friedman, A., Asymptotic estimates for the plasma problem. Duke Math. J. 47 (1980), 705-742.

7. Friedman, A. and Turkington, B., Vortex rings : existence and asymptotic estimates. Trans. Amer. Math. Soc. 268 (1981), 1-37.

8. Gallouët, T., Quelques résultats sur une équation appraisement en physique des plasma. C.R. Acad. Sci. Paris 286 (1978), 739-743.

9. Goldshtik, M.A., A mathematical model of separated flows in an incompressible liquid. Soviet Physics - Doklady - 7 (1963), 1090-1093.

10. Keady, G., An elliptic boundary-value problem with a discontinuous nonlinearity. Proc. Roy. Soc. Edinburgh 91A (1981), 161-174.

11. Keady, G. and Norbury, J., A semilinear elliptic eigenvalue problem, I, II. Proc. Roy. Soc. Edinburgh 87A (1980), 65-82, 83-109.

12. Lin, C.C., On the motion of vortices in two dimensions. Univ. of Toronto Studies, Applied Math. Series no. 5, 1943.

13. Morrey, C.B., Multiple integrals in the calculus of variations. Springer-Verlag, 1966.

14. Norbury, J., Steady vortex pairs in an ideal fluid. Comm. Pure Appl. Math. 28 (1975), 679-700.

15. Richardson, S., Vortices, Liouville's equation and the Bergman kernel function. Mathematika 27 (1980), 321-334.

16. Saffman, P.G. and Baker, G.R., Vortex interactions. Annual Review of Fluid Mechanics Vol. 11, 1979.

17. Serrin, J., A symmetry problem in potential theory. Arch. Rational Mech. Anal. 43 (1971), 304-318.

BT/jgb

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 2406	2. GOVT ACCESSION NO.: AD-A120 965
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): On Steady Vortex Flow in Two Dimensions, I	5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Bruce Turkington	8. CONTRACT OR GRANT NUMBER(s): DAAG29-80-C-0041, MCS-8002927
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE: July 1982
	13. NUMBER OF PAGES: 31
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office): See Item 18	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P.O. Box 12211
Research Triangle Park,
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Vorticity, variational inequality, asymptotic analysis, singular limit, point vortex, free boundary problem

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We study a family of steady flows of an ideal fluid in a bounded domain $D \subseteq R^2$ using a variational method. Adapting an approach due to Arnold, we characterize dynamically possible flows as constrained extremals of the kinetic energy considered as a functional of the vorticity $\omega(x)$, $x \in D$. We restrict our attention to solutions having the special form:

ABSTRACT (Continued)

$$\omega = \omega_\lambda = \lambda\, I_{\{\psi > 0\}}\ ,\ \int_D \omega_\lambda dx = 1$$

where λ is a free parameter and ψ (the streamfunction) is defined by $-\Delta\psi = \omega$ in D, $\psi = -\mu$ on ∂D (μ is an undetermined positive constant). A simple proof of existence of solutions is provided by a direct variational method. The main results concern the singular limit $\lambda \to \infty$ (which implies $\mu \to \infty$). We prove that $\omega_\lambda \to \delta_{x^*}$, the Dirac delta measure at $x^* \in D$, in the distributional sense, and we characterize the point x^* in terms of the geometry of D. Further, we show that supp ω_λ tends asymptotically (in a strong sense) to an infinitesimal disc about x^*. This follows since appropriately scaled version of the solutions ω_λ tend to the unique radial solution of a corresponding limit problem. In fluid dynamical terms, the classical point vortex is obtained as the limit of vortices of finite cross-section (of the Rankine type).